I.S.I. Bangalore Centre
Mid-semestral Examination 2001-2002
B. Math. Hons.II year - Algebra III
21.09.2001 — 9.30 - 12.30 — Instructor : B.Sury

Part A : 60 minutes

A 1.
(a) *Define a finitely generated module over a ring A. When is it called torsion-free?*
(b) *If M is an A-module and $S \subset M$ a subset of it, when is M said to be free on the set S?*

A 2.
State the structure theorem for finitely generated modules over a PID.

A 3.
Let L/K be a field extension.
(a) *Define the degree of L over K.*
(b) *If $\alpha \in L$, when is it said to be algebraic over K?*
(c) *If $\alpha \in L$, define $K[\alpha]$ and $K(\alpha)$.*
(d) *If $[L : K] = 17$, how many fields E exist with $K \subseteq E \subseteq L$.*

A 4.
If $\alpha \in L$ is algebraic over K, define its minimal polynomial $p \in K[X]$ over K and show that if $f \in K[X]$ with $f(\alpha) = 0$, then $f = pq$ for some $q \in K[X]$.

A 5.
If $f \in K[X]$ is monic, irreducible, prove that $L = K[X]/(f)$ is a finite extension of K in which f has a root.

A 6.
Define algebraic closure of a field.
(a) *Use Eisenstein's criterion to the polynomials $X^n - 2$ as n varies to produce an infinite subset of $\overline{\mathbf{Q}}$, the algebraic closure of $\mathbf{Q}$ which is linearly independent over $\mathbf{Q}$.*
(b) *Is $\mathbf{C}(X)$ algebraically closed ? Why ?*

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<u>Part B : 90 minutes</u>

B 1.
If $\sigma : K \to K'$ is a field isomorphism, and $f \in K[X]$ is monic, irreducible, prove that $f^\sigma \in K'[X]$ is monic, irreducible. Further, if L, L' are extensions of K, K' respectively, containing roots α, α' of f, f^σ, show that σ extends to an isomorphism from $K(\alpha)$ to $K'(\alpha')$.

B 2.
Let L/K be a field extension. If $\alpha, \beta \in L$ are algebraic over K, prove that $\alpha + \beta$ is algebraic over K.

B 3.
If Ω is an algebraic closure of a field K, prove that $\alpha, \beta \in \Omega$ are conjugates if, and only if, they have the same minimal polynomial over K.

B 4.
Let G be any group and K be any field. Show that the set of group homomorphisms from G to K^* is a K-linearly independent set i.e., if $\phi_1, \cdots, \phi_r : G \to K^*$ and $a_1, \cdots, a_r \in K$ satisfy $\sum_{i=1}^r a_i \phi(g) = 0$ for all $g \in G$, then $a_i = 0$ for all i.

B 5.
Compute the group of automorphisms of $\mathbf{C}(X)$ which restrict to the identity on $\mathbf{C}$.

B 6.
Let L/K be a (possibly infinite) algebraic extension which is separable. Suppose there exists n such that for any $a \in L$, the degree of $min(a, K)$ is $\leq n$. Prove that L is a finite extension of K and $[L : K] \leq n$.